Potential-Density Basis Sets for Galactic Disks

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ABSTRACT

A class of complete potential-density basis sets in cylindrical (R, ϕ, z) coordinates is presented. This class is suitable for stability studies of galactic disks in three dimensions and includes basis sets tailored for disks with vertical density profiles that are exponential ($e^{-|z|/z_0}$), Gaussian ($e^{-(z/z_0)^2}$) or locally isothermal ($\mathrm{sech}^2(z/z_0)$). The basis sets are non-discrete and non-biorthogonal; however, the extra numerical computations required (compared with discrete biorthogonal sets) are explained and constitute a small overhead. The method of construction (and proof of completeness) is simple and can be used to construct basis sets for other density distributions that are best described in circular or elliptic cylindrical coordinates. When combined with a basis set designed for spheroidal systems, the basis sets presented here can be used to study the stability of realistic disks embedded in massive halos.

Subject headings: galaxies: kinematics and dynamics — galaxies: structure — instabilities — methods: numerical — celestial mechanics: stellar dynamics

1. Introduction

The first step in building a galaxy model is to find or approximate the potential ψ generated by a model mass density ρ. Poisson's equation,

$$\nabla^2 \psi = 4\pi G \rho \,, \tag{1.1}$$

can be solved using basis sets of potential-density pairs $\{(\psi_j, \rho_j) : \nabla^2 \psi_j = 4\pi G \rho_j\}$. A given density ρ is expanded in the basis density functions,

$$\rho = \sum_j c_j \rho_j \,; \tag{1.2a}$$

since Eq. (1.1) is linear, the corresponding potential is

$$\psi = \sum_j c_j \psi_j \,, \tag{1.2b}$$

with the same coefficients c_j. An approximate solution is obtained using finitely many terms.

In this way we can approximate the potentials (and force fields) of mass distributions that are not amenable to an exact analytical solution of Eq. (1.1). Moreover, basis expansions are fundamental to semi-analytical normal mode analyses of stellar systems (Kalnajs 1977) and to a powerful N-body simulation technique (Clutton-Brock 1972) that is ideally suited to stability studies (Earn & Sellwood 1995). The first step required before implementing these methods is to find a basis set that is well-suited to the model of interest.

The eigenfunctions of the Laplacian operator ∇^2 always form a complete biorthogonal basis set (*e.g.*, Courant & Hilbert 1953, §6.3; Arfken 1985, §9.4) but they do not always converge sufficiently fast to typical potentials relevant for galactic dynamics. It is essential that a given model and its normal modes can be represented with a modest number of basis functions; otherwise, accurate expansions become prohibitively expensive.

A variety of useful sets have been derived for flat disks (Clutton-Brock 1972; Kalnajs 1976; Qian 1992, 1993) and for spheroidal systems (Clutton-Brock 1973; Hernquist & Ostriker 1992; Saha 1993; Syer 1995; Robijn & Earn 1996; Zhao 1996). However, the literature has apparently been void of sets that are well-suited to disks of finite thickness. Such basis sets are needed for studies of realistic three-dimensional (3D) disk galaxy models.

There is no evidence that the vertical structure of galactic disks varies significantly with radius. The goal of the present paper is therefore to find basis sets suitable for studies of 3D disks with mass densities of the separable form

$$\rho(R, z) = \rho_R(R)\, \rho_z(z) \ . \tag{1.3}$$

Observations indicate that the distribution of luminous mass in disk galaxies is well-approximated by (1.3) with

$$\rho_R(R) = \frac{M}{2\pi a^2}\, e^{-R/a} \ , \qquad \rho_z(z) = \frac{1}{2b}\, \mathrm{sech}^2(z/b) \ , \tag{1.4}$$

or with $\rho_z(z) \propto e^{-|z|/b}$ or some other power of $\mathrm{sech}(z/b)$ (*cf.* Freeman 1970; van der Kruit 1988). The vertical density profile $\rho_z(z) \propto \mathrm{sech}^2(z/b)$ is favoured theoretically because it results if we demand that the disk be locally isothermal (Spitzer 1942; Binney & Tremaine 1987, problem 4-25). In (1.4), M is the total mass and a and b are scale lengths.

New basis sets for 3D disks are derived in this paper (some of the main results were summarized briefly by Earn 1995) and a simple proof of completeness is also provided. A derivation of the Laplacian eigenfunctions is reviewed below; the present approach is essentially a modification of this procedure, together with a simple trick. Different methods for constructing basis sets for 3D disks are discussed by Robijn & Earn (1996).

2.   The Standard Basis

As a first step, we derive the set of eigenfunctions for the Laplacian operator in cylindrical (R, ϕ, z) coordinates. Thus we seek solutions of

$$\nabla^2 \psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = \lambda\psi \ . \tag{2.1}$$

This will give potential-density pairs that are proportional ($\rho = \frac{\lambda}{4\pi G}\psi$). Separating variables,

$$\psi(R, \phi, z) = \mathcal{R}(R)\, \Phi(\phi)\, \mathcal{Z}(z) \ , \tag{2.2}$$

we obtain from Eq. (2.1) the three ordinary differential equations,

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\mathcal{R}}{dR}\right) + \left(k^2 - \frac{m^2}{R^2}\right)\mathcal{R} = 0 \ , \tag{2.3a}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \ , \tag{2.3b}$$

$$\frac{d^2\mathcal{Z}}{dz^2} - \left(k^2 + \lambda\right)\mathcal{Z} = 0 \ , \tag{2.3c}$$

where k and m are separation constants; k is real and can be taken positive while m is an integer due to periodicity of the ϕ coordinate. These equations (2.3) have the well-known solutions

$$\mathcal{R}(R) = J_m(kR) \ , \tag{2.4a}$$

$$\Phi(\phi) = e^{im\phi} \ , \tag{2.4b}$$

$$\mathcal{Z}(z) = e^{\pm(k^2+\lambda)^{1/2} z} \ , \tag{2.4c}$$

where J_m is the cylindrical Bessel function of order m.

It is easy to see that the eigenvalue λ must be negative [multiply Eq. (2.1) by ψ and integrate by parts using Green's theorem (*e.g.*, Arfken 1985, eq. 1.98) to get $-\int(\nabla\psi)^2\,dV = \lambda\int\psi^2\,dV$]. For our purpose of obtaining a basis set, we can impose the further restriction that $\lambda \leq -k^2$. Thus we may write $\mathcal{Z}(z) = e^{ihz}$, where h is any real number; the eigenvalue is then $\lambda = -(k^2 + h^2)$. The eigenfunctions are

$$\psi_{kmh}(R,\phi,z) = J_m(kR)\,e^{im\phi}\,e^{ihz} \ . \tag{2.5a}$$

To each eigenpotential corresponds the density

$$\rho_{kmh}(R,\phi,z) = -\frac{k^2+h^2}{4\pi G}\,J_m(kR)\,e^{im\phi}\,e^{ihz} \ . \tag{2.5b}$$

By standard theorems (*e.g.*, Courant & Hilbert 1953, §6.3; Arfken 1985, §9.4) these functions form a complete, orthogonal basis for the space of square-integrable functions, $\mathbb{L}^2(\mathbb{R}^3)$. Note that to obtain correct units explicitly, a factor GM/L should be appended to $J_m(kR)$, where L and M are length and mass scales.

The standard basis of eigenfunctions (2.5) is very simple, but it is poorly suited for numerical work with density distributions that are nearly flat, like disk galaxies. In particular, individual basis functions do not satisfy the natural galactic boundary condition, $\psi \to 0$ as $R^2 + z^2 \to \infty$. At the very least we need basis functions that decay both as $R \to \infty$ and as $z \to \pm\infty$.

3. Basis functions for galactic disks

We can derive more suitable basis sets by relaxing the conditions we imposed in Eqs. (2.1) and (2.2). Rather than demanding eigenfunctions, we shall specify the vertical density profile $\rho_z(z)$ in advance and seek solutions of Poisson's equation (1.1) with

$$\psi(R, \phi, z) = \mathcal{R}(R)\, \Phi(\phi)\, \mathcal{Z}(z) \, , \tag{3.1a}$$

$$\rho(R, \phi, z) = \frac{1}{4\pi G}\, \mathcal{R}(R)\, \Phi(\phi)\, \rho_z(z) \, . \tag{3.1b}$$

Note the difference between this and ordinary separation of variables: here we are free to choose the form of $\rho_z(z)$ and we will not have $\mathcal{Z}(z) \propto \rho_z(z)$, unless we choose $\rho_z(z) = e^{ihz}$.

In this case, Eq. (1.1) again separates into three ordinary differential equations, namely Eqs. (2.3a) and (2.3b) together with

$$\frac{d^2 \mathcal{Z}}{dz^2} - k^2 \mathcal{Z} = \rho_z(z) \, , \tag{3.2}$$

which replaces Eq. (2.3c).

We are interested in bell-shaped density factors $\rho_z(z)$ that resemble actual vertical density profiles of disk galaxies, and we impose the boundary conditions that the potential factor $\mathcal{Z}(z) \to 0$ as $z \to \pm\infty$. Many analytical solutions of this form can be found for Eq. (3.2). Several simple and useful solutions are listed in Table 3.1. The first row gives Green's function for Eq. (3.2); when inserted in Eq. (3.1) we obtain the well-known Bessel function basis set for flat disks, used by Toomre (1981) for his flat disk stability analysis. Rows 2–4 give examples that allow us to construct 3D basis sets for realistic disk galaxy models.

The density factors $\rho_z(z)$ in Table 3.1 do not depend on the radial index k. As a result, the full 3D potential of any disk of the form (1.3) with $\rho_z(z)$ drawn from Table 3.1 can be obtained by a single integration over k, provided the Hankel transform of the radial density profile $\rho_R(R)$ is known analytically:

$$\psi(R, z) = \rho_z(z) \int_0^\infty J_0(kR)\, \mathcal{Z}(z)\, S(k)\, dk \, , \tag{3.3}$$

where $S(k)$ is the Hankel transform of $\rho_R(R)$. In this context, the solutions in Table 3.1 have been found previously (e.g., Casertano 1983, Kuijken & Gilmore 1989, Sackett &

Table 3.1: Useful solutions of Eq. (3.2)

Density factor $\rho_z(z)$	Potential factor $\mathcal{Z}(z)$
$\delta(z)$	$e^{-k|z|}$
$e^{-\alpha|z|}$	$\begin{cases} -\frac{1}{k^2-\alpha^2}\left(e^{-\alpha|z|} - \frac{\alpha}{k}e^{-k|z|}\right), & k \neq \alpha \\[2em] -\frac{1}{2k^2}\left(1 + k\,|z|\right)e^{-k|z|}, & k = \alpha \end{cases}$
$e^{-\alpha^2 z^2}$	$-\frac{e^{(k/2\alpha)^2}\sqrt{\pi}}{4\,k\,\alpha}\left[e^{kz}\,\mathrm{erfc}(\frac{k}{2\alpha}+\alpha z) + e^{-kz}\,\mathrm{erfc}(\frac{k}{2\alpha}-\alpha z)\right]$
$\mathrm{sech}^q(\alpha z)$	$-\frac{2^{q-1}}{k^2+q\,k\,\alpha}\left[e^{q\alpha z}{}_2F_1(q, \frac{q}{2}+\frac{k}{2\,\alpha}; 1+\frac{q}{2}+\frac{k}{2\,\alpha}; -e^{2\,\alpha\,z})\right.$ $\left. + e^{-q\alpha z}{}_2F_1(q, \frac{q}{2}+\frac{k}{2\,\alpha}; 1+\frac{q}{2}+\frac{k}{2\,\alpha}; -e^{-2\,\alpha\,z})\right]$

Sparke 1990, Cudderford 1993). The principal contribution of this paper is to show that these solutions, and simpler solutions given below, can be used to construct complete 3D basis sets.

If we are willing to let $\rho_z(z)$ itself depend on the radial wave number k then we can find somewhat simpler solutions from which to construct complete basis sets (Table 3.2). Several basis functions for each k are then required to represent a fixed vertical density profile such as $\mathrm{sech}^2(\alpha z)$. However, this may impose no practical disadvantage for applications where the system evolves and/or where only the large-scale normal modes of the model are required.

Table 3.2: Simpler Solutions with $\rho_z(z)$ depending on k

Density factor $\rho_z(z)$	Potential factor $\mathcal{Z}(z)$
$k^2\,e^{-k|z|}$	$-\frac{1}{2}(1+k\,|z|)\,e^{-k|z|}$
$k^2\,e^{-(kz)^2}$	$-\frac{1}{4}e^{1/4}\sqrt{\pi}\,[e^{kz}\,\mathrm{erfc}(1/2+kz)+e^{-kz}\,\mathrm{erfc}(1/2-kz)]$
$k^2\,\mathrm{sech}(kz)$	$kz\,e^{kz}-\cosh{(kz)}\log{(1+e^{2kz})}$
$k^2\,\mathrm{sech}^2(kz)$	$1+\sinh(kz)\,\arctan[\sinh(kz)]-\frac{\pi}{2}\cosh(kz)$

4. New basis sets

So far we have merely found potential-density pairs; we do not yet have 3D basis sets.

4.1. Construction

To form basis sets from the potential-density pairs indicated in Table 3.1 and Table 3.2, we make the following observation: the left hand side of Eq. (3.2) is invariant under the translation $z \to (z-h)$ for any h, so if we replace $\rho_z(z)$ by $\rho_z(z-h)$ and $\mathcal{Z}(z)$ by $\mathcal{Z}(z-h)$ then we have another potential-density pair. This corresponds to shifting the original configuration up a distance h. As will be shown in the next subsection, if we let h vary from $-\infty$ to ∞ then the functions

$$\psi_{kmh}(R,\phi,z) \;=\; J_m(kR)\,e^{im\phi}\,\mathcal{Z}(z-h)\;, \tag{4.1a}$$

$$\rho_{kmh}(R,\phi,z) \;=\; \frac{1}{4\pi G}\,J_m(kR)\,e^{im\phi}\,\rho_z(z-h)\;, \tag{4.1b}$$

form an $\mathbb{L}^2$-complete basis, where $k \in \mathbb{R}^{>0}$, $m \in \mathbb{Z}$, and $\mathcal{Z}(z)$ and $\rho_z(z)$ are taken from any row in Table 3.1 or Table 3.2. Representing a given model amounts to weighting and stacking the functions (4.1) along the z-axis.

4.2. Completeness

It is sufficient to show that any member of the standard basis (2.5) can be represented. Since our sets differ from the standard basis only in their vertical factors $\rho_z(z)$, it is enough to prove that for any $h \in \mathbb{R}$ there exists a function $w_h(h')$ such that

$$\int_{-\infty}^{\infty} w_h\left(h'\right) \rho_z\left(z - h'\right) dh' = e^{ihz} . \tag{4.2}$$

This Fredholm integral equation of the first kind can be solved explicitly analytically (*cf.* Titchmarsh 1937). Taking the Fourier transform of Eq. (4.2) and using the convolution theorem, we have

$$\tilde{w}_h\left(u\right) \tilde{\rho}_z\left(u\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ihz} e^{iuz} dz = \delta\left(u + h\right) , \tag{4.3}$$

where $\tilde{w}_h(u)$ and $\tilde{\rho}_z(u)$ denote the Fourier transforms of $w_h(h')$ and $\rho_z(h')$ respectively, and δ is the Dirac delta distribution. It follows immediately that

$$w_h\left(h'\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\delta\left(u + h\right)}{\tilde{\rho}_z\left(u\right)} e^{-iuh'} du = \frac{1}{\sqrt{2\pi}} \frac{e^{ihh'}}{\tilde{\rho}_z\left(-h\right)} . \tag{4.4}$$

This formula is meaningful provided $\rho_z(z)$ has a Fourier transform that is nowhere zero. Table 4.1 gives the Fourier transforms of all the functions in Table 3.1 and Table 3.2 [except for arbitrary powers of $\text{sech}(\alpha z)$; for that transform see Oberhettinger 1973, §I.7.211]. Each of these is strictly positive, so our new basis sets are complete in $\mathbb{L}^2(\mathbb{R}^3)$.

Table 4.1: Fourier transforms

Density factor $\rho_z(z)$	$\tilde{\rho}_z(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho_z(z) e^{iuz} dz$
$\delta(z)$	$\frac{1}{\sqrt{2\pi}}$
$e^{-\alpha|z|}$	$\left(\frac{2}{\pi}\right)^{1/2} \frac{\alpha}{\alpha^2 + u^2}$
$e^{-(\alpha z)^2}$	$\frac{1}{\sqrt{2}\alpha} e^{-u^2/4\alpha^2}$
$\text{sech}(\alpha z)$	$\frac{1}{2} \text{sech}(\pi u/2\alpha)$
$\text{sech}^2(\alpha z)$	$\frac{\pi}{4\alpha} u \, \text{csch}(\pi u/2\alpha)$

4.3. Expansion coefficients

The inner product of two potential-density pairs is defined by minus the interaction potential energy of the two densities ρ and ρ',

$$
\begin{aligned}
\langle \rho \,|\, \psi' \rangle &\equiv -\frac{L}{GM^2} \int \rho^* \psi' \, dV \\
&= -\frac{L}{4\pi G^2 M^2} \int \left(\nabla^2 \psi^* \right) \psi' \, dV \ .
\end{aligned}
\tag{4.5}
$$

Here, the asterisk denotes complex conjugation and the factor L/GM^2 makes the inner product dimensionless. All functions are assumed to live in $I\!\!L^2(\mathbb{R}^3)$. In cylindrical coordinates,

$$
\langle \rho \,|\, \psi' \rangle = - \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \rho^* \psi' \, R \, dR \, d\phi \, dz \,,
\tag{4.6}
$$

where we have set $G = M = L = 1$ for convenience.

A basis is a linearly independent set that spans the full vector space of potential-density pairs. Thus for any basis set $\{(\psi_j, \rho_j) : \nabla^2 \psi_j = 4\pi \rho_j\}$ and any given density $\rho \in I\!\!L^2(\mathbb{R}^3)$ there is a set of complex coefficients $\{c_j\}$ such that $\rho = \sum_j c_j \rho_j$. It follows immediately that $\langle \rho \,|\, \psi_{j'} \rangle = \sum_j c_j \langle \rho_j \,|\, \psi_{j'} \rangle$, so defining

$$
\mathcal{M}_{jj'} \equiv \langle \rho_j \,|\, \psi_{j'} \rangle \,,
\tag{4.7}
$$

we have

$$
c_j = \sum_{j'} \mathcal{M}_{jj'}^{-1} \langle \rho \,|\, \psi_{j'} \rangle \,,
\tag{4.8}
$$

where $\mathcal{M}_{jj'}^{-1}$ is the jj' element of matrix inverse of $\mathcal{M}$. The potential generated by ρ is $\psi = \sum_j c_j \psi_j$. A basis is said to be *biorthogonal* if $\mathcal{M}_{jj'}$ is diagonal, *i.e.*, $\langle \rho_j \,|\, \psi_{j'} \rangle \neq 0$ if and only if $j = j'$, and *biorthonormal* if $\mathcal{M}_{jj'}$ is the identity matrix $\delta_{jj'}$. With a biorthonormal basis, the dimensionless expansion coefficients are simply $c_j = \langle \rho \,|\, \psi_j \rangle = \langle \rho_j \,|\, \psi \rangle$. Any finite subset of a basis can be made biorthonormal with the Gram-Schmidt algorithm (*e.g.*, Arfken 1985). Biorthogonality is clearly preferable, but without it the only extra effort required is the one-time evaluation and inversion of the matrix $\mathcal{M}_{jj'}$ (4.7), as emphasized by Saha (1993).

In the present context, where the abstract index j represents the triplet (k, m, h) with k and h real, the sum over j refers to a discrete sum and two integrals,

$$
\rho = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} c_{kmh} \, \rho_{kmh} \, dk \, dh \,,
\tag{4.9a}
$$

$$\psi \;=\; \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} c_{kmh}\,\psi_{kmh}\,dk\,dh \;. \tag{4.9b}$$

The matrix $\mathcal{M}_{jj'}$ refers to

$$\mathcal{M}_{kmh\,k'm'h'} = \langle \rho_{kmh} \,|\, \psi_{k'm'h'} \rangle \;, \tag{4.10}$$

and biorthonormality means strictly that

$$\mathcal{M}_{kmh\,k'm'h'} = \delta\left(k - k'\right)\,\delta_{mm'}\,\delta\left(h - h'\right) \;. \tag{4.11}$$

Using the Bessel closure relation,

$$\int_{0}^{\infty} J_m\left(kR\right) J_m\left(k'R\right)\,R\,dR = \frac{1}{k}\,\delta\left(k - k'\right) \;, \tag{4.12}$$

(*e.g.*, Arfken 1985, Eq. 11.59) and the exponential identity,

$$\int_{0}^{2\pi} e^{-im\phi} e^{im'\phi}\,d\phi = 2\pi\delta_{mm'} \;, \tag{4.13}$$

we find for the present class of basis sets (4.1),

$$\mathcal{M}_{kmh\,k'm'h'} = \delta\left(k - k'\right)\,\delta_{mm'}\,\mathcal{M}_{khh'} \;, \tag{4.14}$$

where

$$\mathcal{M}_{khh'} \equiv -\frac{1}{2k} \int_{-\infty}^{\infty} \rho_z^*\left(z - h\right) \mathcal{Z}\left(z - h'\right)\,dz \;. \tag{4.15}$$

The expansion coefficients are therefore

$$\begin{aligned}
c_{kmh} &= \sum_{m'=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathcal{M}_{kmh\,k'm'h'}^{-1} \,\langle \rho \,|\, \psi_{k'm'h'} \rangle\,dk'\,dh' \\
&= \int_{-\infty}^{\infty} \mathcal{M}_{khh'}^{-1}\,\langle \rho \,|\, \psi_{kmh'} \rangle\,dh' \;,
\end{aligned} \tag{4.16}$$

where $\mathcal{M}^{-1}$ is the matrix inverse of $\mathcal{M}$. To find $\mathcal{M}_{khh'}^{-1}$ we must solve the integral equation

$$\int_{-\infty}^{\infty} \mathcal{M}_{khh''}^{-1} \mathcal{M}_{kh''h'}\,dh'' = \delta\left(h - h'\right) \;. \tag{4.17}$$

We discuss how to solve this equation numerically in the next section.

5. Discretization

We have given several crucial formulae that involve integrals over the radial and vertical basis indeces, k and h [(4.9), (4.16), (4.17)]. Since our goal is to use as few basis functions as possible to obtain a given accuracy, these integrations must be reduced to computations with a small, finite number of k and h values.

5.1. Numerical quadrature

The continuous indeces k and h can be replaced with discrete indeces n and ℓ via prescriptions of the form

$$\int_0^\infty f(k)\ dk \ \longrightarrow\ \sum_{n=0}^N w_n^k f(k_n)\,, \tag{5.1a}$$

$$\int_{-\infty}^\infty g(h)\ dh \ \longrightarrow\ \sum_{\ell=0}^L w_\ell^h g(h_\ell)\,, \tag{5.1b}$$

where the sums approximate the integrals to some given order in $1/N$ and $1/L$. The simplest approach is to choose a set of evenly spaced abscissae, $k_n = n\Delta k$ and $h_\ell = (\ell - L/2)\Delta h$, and to use a classical formula such as Simpson's rule to define the weights (*e.g.*, Abramowitz & Stegun 1972, §25.4.6; Press *et al.* 1992, Eq. 4.1.13). In the case of the axisymmetric ($m = 0$) integrals over k in Eq. (4.9), we can do better by employing a Gaussian quadrature rule (*e.g.*, Press *et al.* 1992, §4.5) because we know the initial radial profile.

5.2. Solving the integral equation for $\mathcal{M}^{-1}$

In any application, our first numerical task is to solve Eq. (4.17) for $\mathcal{M}_{khh'}^{-1}$. To reduce notational complexity, let us define

$$\mathcal{A}_{\ell\ell'}^n \equiv \mathcal{M}_{knh_\ell h_{\ell'}}\,, \qquad \mathcal{B}_{\ell\ell'}^n \equiv \mathcal{M}_{knh_\ell h_{\ell'}}^{-1}\,. \tag{5.2}$$

We wish to solve for the matrices $\mathcal{B}^n$, $n = 0, \ldots, N$. Using (5.1), Eq. (4.17) becomes

$$\sum_{\ell''=0}^L w_{\ell''}^h \mathcal{B}_{\ell\ell''}^n \mathcal{A}_{\ell''\ell'}^n = 0\,, \qquad \ell \neq \ell'\,, \tag{5.3a}$$

$$w_\ell^h \sum_{\ell''=0}^L w_{\ell''}^h \mathcal{B}_{\ell\ell''}^n \mathcal{A}_{\ell''\ell}^n = 1\,, \tag{5.3b}$$

where the second line is obtained by integrating Eq. (4.17) in a small interval about h_ℓ for $\ell = \ell'$. Putting these two pieces together we have

$$\sum_{\ell''=0}^{L} w_{\ell''}^h \mathcal{B}_{\ell\ell''}^n \mathcal{A}_{\ell''\ell'}^n = \frac{1}{w_\ell^h} \delta_{\ell\ell'} . \tag{5.4}$$

Thus $\mathcal{B}^n$ is the ordinary matrix inverse of $\mathcal{A}^n \mathcal{W}$, where $\mathcal{W} = \text{diag}(w_0^h, \ldots, w_L^h)$. Each matrix $\mathcal{B}^n$ needs to be computed only once for a given basis set, so the work involved is always negligible.

5.3. Discrete expansion formulae

We can now rewrite our formulae for the expansion coefficients and expanded potential in a completely discrete manner. Eq. (4.16) becomes

$$c_{nm\ell} = \sum_{\ell'} w_{\ell'}^h \, \mathcal{B}_{\ell\ell'}^n \, \langle \rho \, | \, \psi_{nm\ell'} \rangle . \tag{5.5}$$

Eq. (4.9b) becomes

$$\psi = \sum_m \sum_\ell w_\ell^h \sum_n w_n^k \, c_{nm\ell} \, \psi_{nm\ell} , \tag{5.6}$$

and a similar formula replaces Eq. (4.9a). Note that in the case of a distribution of point particles, $\rho = \sum_i M_i \, \delta(\boldsymbol{x} - \boldsymbol{x}_i)$, the inner product in (5.5) also becomes a discrete sum,

$$\langle \rho \, | \, \psi_{nm\ell} \rangle = - \sum_i M_i \, \psi_{nm\ell} \, (\boldsymbol{x}_i) . \tag{5.7}$$

6. Discussion

A similar approach can be used to derive basis sets for 3D elliptic disks. In fact, all we need to do is replace the factor $J_m(kr) \, e^{im\phi}$ of our basis sets with the eigenfunctions of the Laplacian in elliptic coordinates (Mathieu functions, *e.g.*, Abramowitz & Stegun 1972, chapter 20). Such basis sets should be useful for stability studies of realistically thickenned versions of flat elliptic disks (*e.g.*, Evans & de Zeeuw 1992).

For stability work, it is essential to be able to represent the *normal modes* of the system with a small number of terms of the chosen basis expansion. There is no guarantee that a basis selected for its ability to represent the unperturbed model will be ideal for

representing its normal modes. It is prudent, therefore, to repeat stability calculations with several different basis sets. Robijn & Earn (1996) discuss alternatives to the present class.

Both in stability analyses and N-body experiments, only the basis potentials are used for the principal computations: in Eqs. (5.5), (5.6) and (5.7) the basis densities do not appear. For this reason, it is unfortunate that in Tables 3.1 and 3.2 the potential functions are more complicated than the associated densities. Nevertheless, they are easy to implement because the special functions they involve are available in all standard numerical libraries. Moreover, all the potentials and densities discussed in this paper are separable so they can be interpolated efficiently and accurately from one-dimensional tables. The basis sets provided here should, therefore, be useful for stability studies and modeling of disk galaxies.

References

Abramowitz M, Stegun IA, 1972, *Handbook of Mathematical Functions,* (Dover: New York)

Arfken G, 1985, *Mathematical Methods for Physicists,* Third Edition (Academic Press)

Binney JJ, Tremaine S, 1987, *Galactic Dynamics* (Princeton University Press)

Casertano S, 1983, *Mon. Not. R. astr. Soc.* **203**, 735–747

Clutton-Brock M, 1972, *Astrophys. and Space Sci.* **16**, 101–119

Clutton-Brock M, 1973, *Astrophys. and Space Sci.* **23**, 55–69

Courant R, Hilbert D, 1953, *Methods of Mathematical Physics,* Vol. 1 (Interscience: New York)

Cudderford P, 1993, *Mon. Not. R. astr. Soc.* **262**, 1076–1086

Earn DJD, 1995, in *Unsolved problems of the Milky Way*, Proc. IAU Symp. 169, edited by L. Blitz, in press

Earn DJD, Sellwood JA, 1995, *Astrophys. J.* **451**, 533–541

Evans NW, de Zeeuw PT, 1992, *Mon. Not. R. astr. Soc.* **257**, 152–176

Freeman KC, 1970, *Astrophys. J.* **160**, 811–830

Hernquist L, Ostriker JP, 1992, *Astrophys. J.* **386**, 375–397

Kalnajs AJ, 1972, *Astrophys. J.* **175**, 63–76

Kalnajs AJ, 1976, *Astrophys. J.* **205**, 745–750

Kalnajs AJ, 1977, *Astrophys. J.* **212**, 637–644

Kuijken K, Gilmore G, 1989, *Mon. Not. R. astr. Soc.* **239**, 571–603

Morse PM, Feshbach H, 1953, *Methods of theoretical physics,* (McGraw-Hill: New York)

Oberhettinger F, 1973, *Fourier Transforms of Distributions and Their Inverses: a collection of tables* (Academic Press: New York)

Press WH, Teukolsky SA, Vetterling WT, Flannery BP, 1992, *Numerical Recipes in C,* Second Edition (Cambridge University Press)

Qian EE, 1992, *Mon. Not. R. astr. Soc.* **257**, 581–592

Qian EE, 1993, *Mon. Not. R. astr. Soc.* **263**, 394–402

Robijn FHA, Earn DJD, 1996, *Mon. Not. R. astr. Soc.* **xx**, in press

Sackett P, Sparke L, 1990, *Astrophys. J.* **361**, 408–418

Saha P, 1993, *Mon. Not. R. astr. Soc.* **262**, 1062–1064

Spitzer L, 1942, *Astrophys. J.* **95**, 329–344

Syer D, 1995, *Mon. Not. R. astr. Soc.* **276**, 1009

Titchmarsh EC, 1937, *Introduction to the Theory of Fourier Integrals* (Chelsea: New York)

Toomre A, 1981, in *The Structure and Evolution of Normal Galaxies,* edited by S.M. Fall and D. Lynden-Bell (Cambridge University Press) pp. 111–136

van der Kruit PC, 1988, *Astron. Astrophys.* **192**, 117-127

Zhao HS, 1996, *Mon. Not. R. astr. Soc.* **xx**, in press